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HARMONICALLY EXCITED ORBITAL VARIATIONS

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ABSTRACT

Rephrasing the equations of motion for orbital maneuvers in terms of Lagrangian generalized coordinates instead of Newtonian rectangular cartesian coordinates can make certain harmonic terms in the orbital angular momentum vector more readily apparent. In this formulation the equations of motion adopt the form of a damped harmonic oscillator when torques are applied to the orbit in a variationally prescribed manner. The frequencies of the oscillator equation are in some ways unexpected but can nonetheless be exploited through resonant forcing functions to achieve large secular variations in the orbital elements. Two cases are discussed using a circular orbit as the control case: a) large changes in orbital inclination achieved by harmonic excitation rather than one impulsive velocity change, and b) periodic and secular changes to the longitude of the ascending node using both stable and unstable excitation strategies.

The implications of these equations are also discussed for both artificial satellites and natural satellites. For the former, two utilitarian orbits are suggested, each exploiting a form of harmonic excitation.

KEYWORDS

Celestial mechanics; angular momentum; harmonic orbital oscillations; zonal harmonics; tesseral harmonics; Halley's comet.

NOMENCLATURE

A,B,C,a,b	= Arbitrary constants
F	= Force vector
H	= Angular momentum vector
I_{xx}, I_{yy}, I_{zz}	= Orbital moments of inertia
I_{xy}, I_{xz}, I_{yz}	= Orbital products of inertia
I	= Inclination angle of satellite orbit with respect to equatorial plane
i	= $\sqrt{-1}$; base of the imaginary numbers
J_0	= Bessel series coefficient
J	= Legendre series coefficient
k	= Subscript index
M	= Applied moment vector
m	= Satellite mass
p	= Circular period of the orbit
r	= Total orbital radius from the focal point
r_0	= Earth radius
T	= Period of defined frequency
t	= Time
w	= Substitution variable
z	= Complex number
α, β	= Axis coordinates for Mathieu stability map
λ_1, λ_2	= Eigenvalue solution to differential equation
$\dot{\phi}$	= Orbital circular frequency
$\dot{\psi}$	= Orbital precession rate
τ_1, τ_2	= Natural frequency components of harmonic oscillator equation
$\boldsymbol{\omega}$	= Angular velocity vector of satellite in Eulerian coordinate system
$\boldsymbol{\Omega}$	= Angular velocity vector of Eulerian coordinates with respect to Newtonian space-fixed coordinates

SUBSCRIPTS

(01)	Initial condition
(PR)	Precession rate

SUPERSCRIPTS

$(\dot{\ })$	Differentiation with respect to time
$(\bar{\ })$	Constant or steady state portion of indicated variable

INTRODUCTION

When the concept of angular momentum is introduced into the study of orbital motion it is generally for one of two purposes, 1) to assist in obtaining analytical vector solutions to the equations of motion in rectangular coordinates or, 2) to provide a physical explanation for planar motion and the energy integral. But if the time history of the angular momentum vector itself is utilized as a method for deriving the equations of motion in a spherical coordinate description, unexpected insights into the consequences of harmonic terms often result.

Expressed in terms of Eulerian rotations certain natural frequencies other than the usual orbital period appear in the generalized coordinates. These consitutute independent modal arms and can be associated with the classical

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notions of precession and nutation. Furthermore, since the time rate of change of angular momentum is equal to the applied torque on an orbit, one quickly appreciates that a more expedient method of orbital maneuver might be accomplished by a sinusoidal variation of thrust at one of these frequencies rather than a single impulsive burn.

In naturally occurring trajectories, advancement of the line of nodes, or more succinctly, precession, has been recognized as a fundamental outgrowth of a non-spherical gravitational potential field virtually from the first rigorous mathematical description of the central force field problem. Its phase analogue, nutation, has been discussed but more often dismissed in subsequent developments. Ehricke[1] introduces a three axis spherical geometry to describe impulsive velocity change events using a pitch-yaw-roll treatment. However, Ehricke's Eulerian rotation sequence is quite different, more closely resembling in his analysis a commonly used aeroballistic coordinate frame. Hansen[2] developed a perturbation theory for a prize winning 1831 essay on the mutual perturbation (gegenseitige Störungen) of Jupiter and Saturn using a so-called "ideal" coordinate system. (The term ideal applies if a derivative mapping of the direction cosines holds during transformation.) Using a single Hansen function, called "W", rectangular space-fixed coordinates can be mapped into coordinates rigidly fixed in an osculating orbital plane. Hansen chose to solve his equations numerically, however, which tended to disguise the underlying harmonic content in his solutions, as well as the possibility of harmonic perturbative disturbing torques. Musen[3] modified the Hansen/Eulerian description by introducing four parameters to linearize the arguments of angles with respect to time and avoid singularities at high orbital inclination. The four parameters are not independent, but three generalized coordinates do result from their quotient combinations. A subtle linearization occurs during this operation, though, since the analytical solution contains Bessel function coefficients of all harmonic terms. Only the coefficients of J_0 rank are addressed in the Musen solution, which again eliminates harmonic terms of higher period.

A more direct correlation may be found in the work of Newcomb[4]. In his approach Laplace's theory is restructured using eccentric anomaly as the independent variable, and thus making any resulting series converge more rapidly in numerical solutions. In effect he arrives at a spherical solution quite similar to Hansen's. Once again, though, the direction cosine mapping does not include the oscillating portion of the sinusoidal variation of the out-of-plane motion, meaning the transformation is "non-ideal" in Hansen's vocabulary. Nonetheless, Newcomb's theory is a powerful tool for determination of positional information. Applications of his theory to the motions of Uranus and Neptune are still used in nautical almanacs today, nearly ninety years after their first publication.

In the following derivation the equation of motion will be developed by first retaining all precessional and nutational components of the orbital progression. Then a discussion of the disturbing forces, either natural or externally imposed follows and the consequences of certain types of perturbation are explored.

THEORY

To define the coordinate system the x-axis will be prescribed to lie along the line of nodes. The z-axis is defined as being always normal to the orbital plane and the y-axis remains mutually perpendicular to both with the positive direction found by use of the right hand rule. Figure 1 displays the geometry. For the purposes of simplicity only circular orbits with constant period will be discussed in the following derivation, though more complex expressions for period can be substituted into the equations at any point with no loss of generality.

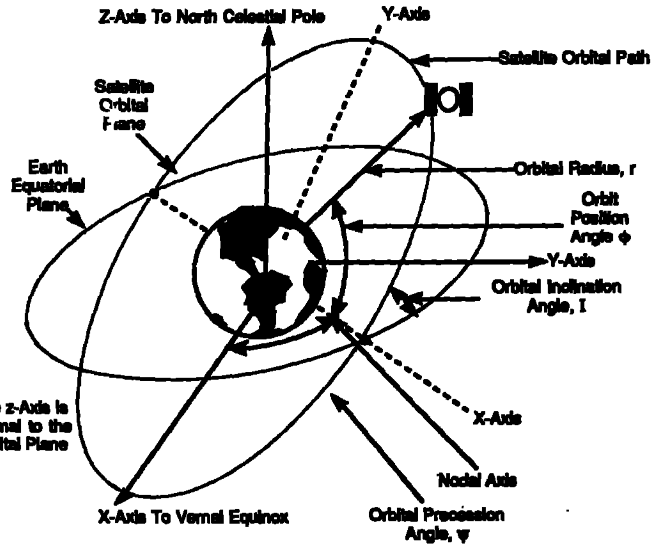

Figure 1 The geometry of the orbit showing the relationship of space fixed Newtonian coordinates (X,Y,Z) with respect to moving Eulerian coordinates (x,y,z)

The angular velocity vector of this system of equations is

$$\boldsymbol{\omega} = (\dot{I},\ \dot{\psi}\sin I,\ \frac{1}{p} + \dot{\psi}\cos I) \tag{1}$$

A strong coupling is thus apparent between the periodic frequency, the advancement of the line of nodes, $\dot{\psi}$, and the orbital inclination I.

The inertial terms about each of the axes are now needed in order to calculate the individual components of angular momentum. These will be called moments and products of inertia in view of their obvious solid body kinetic analogue. They are

$$
\begin{aligned}
I_{xx} &= mr^2\sin^2\dot{\phi}t && \text{a)}\\
I_{yy} &= mr^2\cos^2\dot{\phi}t && \text{b)}\\
I_{zz} &= mr^2 && \text{c) (2)}\\
I_{xy} &= mr^2\sin\dot{\phi}t\cos\dot{\phi}t && \text{d)}\\
I_{xz} &= 0 && \text{e)}
\end{aligned}
$$

The rotational form of the momentum vector is

$$\Sigma \mathbf{M} = \dot{\mathbf{H}} + \boldsymbol{\Omega} \times \mathbf{H} \tag{3}$$

where the angular momentum vector is the product of the inertia tensor and the angular velocity vector. Or

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{4}$$

The time rates of change of the inertial components are:

$$
\begin{aligned}
\dot{I}_{xx} &= mr^2(2\dot{\phi}\cos\dot{\phi}t\sin\dot{\phi}t) && \text{a)}\\
\dot{I}_{yy} &= -\ mr^2(2\dot{\phi}\cos\dot{\phi}t\sin\dot{\phi}t) && \text{b)}\\
\dot{I}_{zz} &= 0 && \text{c) (5)}\\
\dot{I}_{xy} &= mr^2\dot{\phi}(\cos^2\dot{\phi}t - \sin^2\dot{\phi}t) && \text{d)}\\
\dot{I}_{yz} &= 0 && \text{e)}
\end{aligned}
$$

As the coordinate system moves with respect to fixed inertial space it will have angular velocity components of

$$\boldsymbol{\Omega} = (\dot{I},\ \dot{\psi}\sin I,\ \dot{\psi}\cos I) \tag{6}$$

And finally the derivatives with respect to time of the body fixed angular velocity will be

$$\dot{\boldsymbol{\omega}} = (\ddot{I},\ \ddot{\psi}\sin I + \dot{\psi}\,\dot{I}\cos I,\ \ddot{\phi} + \ddot{\psi}\cos I - \dot{\psi}\,\dot{I}\sin I) \tag{7}$$

Substituting equations (4) into equation (3) gives:

$$
\begin{aligned}
\Sigma M_x &= \dot{I}_{xx}\omega_x + I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - \dot{I}_{xy}\omega_y + (\Omega_y H_z - \Omega_z H_y) && \text{a)}\\
\Sigma M_y &= \dot{I}_{yy}\omega_y + I_{yy}\dot{\omega}_y - I_{xy}\dot{\omega}_x - \dot{I}_{xy}\omega_x + (\Omega_z H_x - \Omega_x H_z) && \text{b) (8)}\\
\Sigma M_z &= I_{zz}\,\dot{\omega}_z + \dot{I}_{zz}\omega_z + (\Omega_x H_y - \Omega_y H_x) && \text{c)}
\end{aligned}
$$

While substituting (5), (6) and (7) into (8a) yields:

$$
\begin{aligned}
\Sigma M_x = mr^2(&\dot{I}\ 2\dot{\phi}\cos\dot{\phi}t\sin\dot{\phi}t + \ddot{I}\sin^2\dot{\phi}t\\
&- \dot{\phi}(\cos^2\dot{\phi}t - \sin^2\dot{\phi}t)\ \dot{\psi}\sin I\\
&- \sin\dot{\phi}t\cos\dot{\phi}t(\ddot{\psi}\sin I + \dot{\psi}\,\dot{I}\cos I)\\
&+ \dot{\psi}\sin I(\dot{\phi} + \dot{\psi}\cos I)\\
&+ \dot{\psi}\,\dot{I}\cos I\sin\dot{\phi}t\cos\dot{\phi}t\\
&- \dot{\psi}^2\cos^2\dot{\phi}t\sin I)
\end{aligned} \tag{9}
$$

Performing the same operation about the y-axis gives:

$$
\begin{aligned}
\Sigma \mathbf{M}_y = \mathrm{mr}^2((&-2\dot{\phi}\sin\dot{\phi}t\cos\dot{\phi}t)\dot{\psi}\sin I \\
&+\cos^2\dot{\phi}t(\ddot{\psi}\sin I + \dot{\psi}\,\dot{I}\cos I) \\
&-\dot{\phi}\,\dot{I}\,(\cos^2\dot{\phi}t - \sin^2\dot{\phi}t) \\
&-\ddot{I}\sin\dot{\phi}t\cos\dot{\phi}t \\
&+\dot{\psi}\,\dot{I}\cos I\sin^2\dot{\phi}t \\
&-\dot{\psi}^2\sin I\sin\dot{\phi}t\cos\dot{\phi}t \\
&-\dot{I}\,\dot{\phi} - \dot{I}\,\dot{\psi}\cos I)
\end{aligned}
\tag{10}
$$

And finally in the z-direction:

$$
\begin{aligned}
\Sigma \mathbf{M}_z = \mathrm{mr}^2((&\ddot{\phi} + \ddot{\psi}\cos I - \dot{\psi}\,\dot{I}\sin I) \\
&+(-\dot{I}^2\sin\dot{\phi}t\cos\dot{\phi}t + \dot{\psi}\,\dot{I}\cos^2\dot{\phi}t\sin I) \\
&-(\dot{I}\,\dot{\psi}\sin I\sin^2\dot{\phi}t + \dot{\psi}^2\sin^2 I\sin\dot{\phi}t\cos\dot{\phi}t))
\end{aligned}
\tag{11}
$$

SOLUTION BY SIMPLE QUADRATURE

Equations (9), (10), (11), form a complete set of equations of motion for the total time history of an orbit. Integrating the equations numerically would produce the orbital evolution based on various external or applied torques. But depending on the type of excitation more expedient methods of solution are possible when certain simplifications are considered. Additionally, as will be presently shown, some types of excitation create unstable orbits and in those cases numerical integration actually leads to erroneous results.

The first approximation to consider is the small angle formula in the inclination angle, namely

$$
\begin{aligned}
\sin I &= I \\
\cos I &= 1
\end{aligned}
\tag{12}
$$

Equations (9) and (10) now reduce to:

$$
\begin{aligned}
&\frac{\Sigma \mathbf{M}_x}{\mathrm{mr}^2} = \left(\dot{I}\,\dot{\phi} - \frac{\ddot{\psi}}{2}I\right)\sin 2\dot{\phi}t + \left(-\frac{\ddot{I}}{2} - \dot{\phi}\,\dot{\psi}I - \frac{\dot{\psi}^2 I}{2}\right)\cos 2\dot{\phi}t + \left(\frac{\ddot{I}}{2} + \dot{\psi}\,\dot{\phi}I + \frac{\psi^2}{2}I\right) \qquad \text{a)} \\
&\frac{\Sigma \mathbf{M}_y}{\mathrm{mr}^2} = \left(-\dot{I}\,\dot{\phi} + \frac{\ddot{\psi}}{2}I\right)\cos 2\dot{\phi}t + \left(-\frac{\ddot{I}}{2} - \dot{\phi}\,\dot{\psi}I - \frac{\dot{\psi}^2}{2}I\right)\sin 2\dot{\phi}t + \frac{\ddot{\psi}}{2}I - \dot{\phi}\,\dot{I} \qquad \text{b)}
\end{aligned}
\tag{13}
$$

Rearranging equation (13) into the form of a damped harmonic oscillator and assuming a quasi-steady precession rate, ($\ddot{\psi}=0$);

$$
\frac{\Sigma \mathbf{M}_x}{\mathrm{mr}^2} = (1 - \cos 2\dot{\phi}t)\frac{\ddot{I}}{2} + \dot{\phi}\sin 2\dot{\phi}t\,\dot{I} + \left(\dot{\psi}\,\dot{\phi} + \frac{\dot{\psi}^2}{2}\right)(1 - \cos 2\dot{\phi}t)I \tag{14}
$$

$$
\frac{\Sigma \mathbf{M}_y}{\mathrm{mr}^2} = (-\sin 2\dot{\phi}t)\frac{\ddot{I}}{2} - \dot{\phi}(1 + \cos 2\dot{\phi}t)\,\dot{I} - \left(\dot{\psi}\,\dot{\phi} + \frac{\dot{\psi}^2}{2}\right)(\sin 2\dot{\phi}t)I \tag{15}
$$

Multiplying equation (15) by i and adding to (14) gives

$$
\frac{\Sigma(\mathbf{M}_x + i\mathbf{M}_y)}{\mathrm{mr}^2} = \left(\frac{\ddot{I}}{2} + i\dot{\phi}\,\dot{I} + \left(\dot{\psi}\,\dot{\phi} + \frac{\dot{\psi}^2}{2}\right)I\right)(1 - e^{i2\dot{\phi}t}) \tag{16}
$$

The homogeneous solution to equation (16) can be immediately written in the form

$$
I = I_{01}\,e^{\lambda_1 t} + I_{02}\,e^{\lambda_2 t} \tag{17}
$$

where

$$
\lambda_{1,2} = -i\dot{\phi} \pm \sqrt{(i\dot{\phi})^2 - (2\dot{\psi}\,\dot{\phi} + \dot{\psi}^2)} \tag{18}
$$

Restating the radicand of the eigenvalue in a quadratic form as

$$(i\dot{\phi})^2 - (2\dot{\psi}\dot{\phi} + \dot{\psi}^2) = (-1)(\dot{\phi} + \dot{\psi})^2 \tag{19}$$

resolves the orbit into its component modal arms. Any orbit of such a form exhibits stable harmonic variations in inclination angle whenever the precession rate is prograde, (in the same right hand rule sense as the angular frequency) because the radicand in these cases is always negative. However, if the precession rate is retrograde and greater than the angular velocity, inclination angle is an unstable variable since a positive real root always results from the radicand. The inevitable conclusion is that inclination angle will grow rapidly with time.

In naturally occurring trajectories, the principal mechanism for producing a secular precession rate is the asphericity of the geopotential field. By describing the earth's central force field potential in the usual way as the solution to a Legendre polynomial, then the second zonal harmonic can be identified with a torque that creates precession. By ignoring all terms higher than second order in the Legendre solution a simple expression for the precessional period results:

$$T_{PR} = \frac{T\dot{\phi}}{J\cos i}\left(\frac{r}{r_0}\right)^2 \tag{20}$$

Figure 2 is a plot of equation (20) showing precessional period as a function of orbital inclination for various orbital radii. The precession rate is indeed retorgrade but generally only a few one-hundredths to a few one-thousandths of one percent of the angular frequency. Hence most satellite orbits are stable in inclination angle with two natural frequencies of

$$\tau_1 = 2\dot{\phi} - \dot{\psi}$$
$$\tau_2 = \dot{\psi}$$

Equation (16) is now rendered in a highly suggestive form for finding a method of exciting large inclinations by sinusoidally applied torques at one of the natural frequencies of nutation. Rewriting the coefficient $(1 - e^{i2\dot{\phi}t})$ as $(e^{i\omega t} - e^{i2\dot{\phi}t})$ produces a forcing function of

$$\frac{(M_x + iM_y)}{mr^2(e^{i\omega t} - e^{i2\dot{\phi}t})} = \frac{z}{mr^2}\left(\frac{1}{e^{i\omega t} - e^{i2\dot{\phi}t}}\right) \tag{21}$$

By applying moments harmonically such that

$$z = e^{i(\tau_k - \omega)t} - e^{i(\tau_k - 2\dot{\phi})t} \qquad k = 1,2 \tag{22}$$

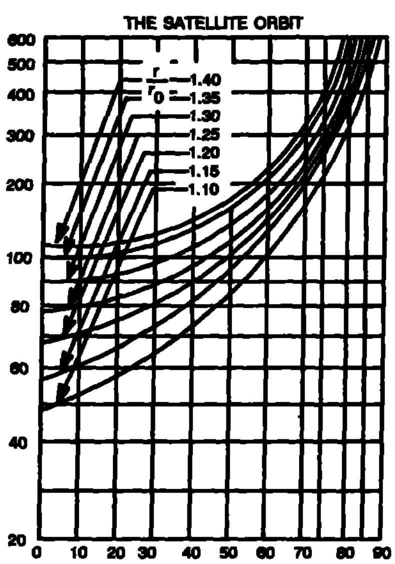

Figure 2 Period of orbital precession as function of orbital inclination (first-order solution; near-circular orbits; r = semi-major axis). (Ehricke)

a resonance solution can be expected from equation (16). An exponential increase in nutation follows logically at least to the extent of the two assumptions of small angles and constant precession rate.

Note that the two roots available from equation (18) are actually precessional conjugate solutions so no minimal or maximal energy expenditure benefit is gained by using one or the other.

OTHER EXCITATION STRATEGIES

For the specific purpose of exploiting an orbital instability one excitation scheme would be to apply torques about the nutational axis proportional to the time rate of change of the product of the inclination velocity and one half of the second harmonic of the orbital period. Or, mathematically,

$$\frac{\Sigma M_x}{mr^2} = \frac{d}{dt}\left(-\frac{1}{2}\cos 2\dot{\phi}t\ \dot{I}\right) \tag{23}$$

Substitution of equation (23) into (14) leads to

$$0 = \ddot{I} + (\alpha - 2\beta\cos 2\dot{\phi}t)I \tag{24}$$

which can immediately be recognized as Mathieu's equation. Figure 3 gives a map of the stability of Mathieu's equation for various values of the coefficients α and β. Any number of specific solutions are available depending on the values of the two Mathieu parameters. Using the problem as stated in equations (23) and (14) a satellite orbit is exactly neutrally stable. Adding only an incremental change in the applied torque proportional to the negative of the orbital inclination, shifts the Mathieu solution until it resides wholly in the unstable region. Now large changes in inclination arise from small thrusts.

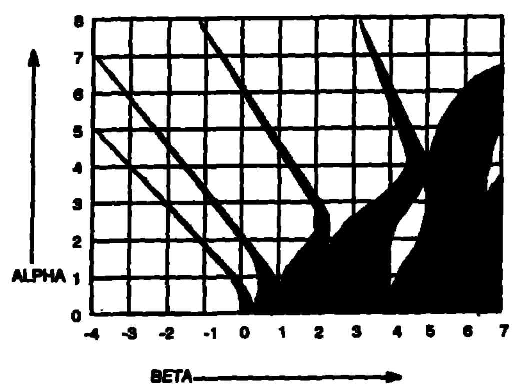

Figure 3 Fundamental diagram determining the stability of a system with variable elasticity. The shaded regions are stable and the blank regions are unstable. (Van der Pol and Strutt.)

For comparison, this approach to increasing orbital inclination uses a proportionally smaller amount of energy than expending force at only one point in an orbit. The price, however, is paid in the time required to effect an orbital change. Depending on the magnitude of the applied torques, an inclination change by an instantaneous burn will be realized within some fraction of the orbital period. Even with an intermediate, highly elliptical, orbit maneuver the total inclination change will be accomplished on the order of one orbital period. On the other hand, change by harmonic excitations requires a few to several tens of orbital periods. In the sense that the total integral of the work needed for an inclination change is not path independent, the system is not conservative in the classical sense. Once it has been disturbed, it requires an equivalent amount of work to restore the system to its initial state as was initially absorbed to disturb it.

Expanding the analysis in the precessional direction the requirement that precession rate remain a constant will now be relaxed.

Multiplying equation (14) by i and adding it directly to equation (13) gives:

$$\frac{\Sigma(M_x + iM_y)}{mr^2} = \left(\frac{\ddot{I}}{2} + i\left(\dot{\phi}\dot{I} + \frac{\ddot{\psi}}{2}I\right) + \left(\dot{\psi}\dot{\phi} + \frac{\dot{\psi}^2}{2}\right)I\right)(1 - e^{i2\dot{\phi}t}) \tag{25}$$

The homogeneous portion of the imaginary section of equation (25) can be written:

$$0 = \dot{\phi}\dot{I} + \frac{\ddot{\psi}}{2}I \tag{26}$$

By a simple substitution of $w = \dot{\psi}$, and separation of variables quadrature, equation (26) can be integrated directly with the result

$$\ln I = C_1 \dot{\psi} \tag{27}$$

Equation (27) demonstrates the strong coupling between all three of the Euler angle coordinates. In fact the former condition of maintaining a constant $\dot{\psi}$ during change can only be maintained if both I and $\dot{\psi}$ are harmonic with identical period, but phase-shifted by 90°.

One final form of the equation shall be invoked. In the case of a constant inclination angle, equation (10) can be written:

$$\frac{\Sigma M_y}{mr^2 \sin I} = \ddot{\psi}(\cos^2 \dot{\phi} t) - \frac{1}{2}(2\dot{\phi}\dot{\psi} + \dot{\psi}^2)\sin 2\dot{\phi} t \tag{28}$$

If moments are applied about the y-axis proportional to the longitude of the ascending node, ψ, when the circular frequency is much greater than the precessional frequency then:

$$\frac{\Sigma M_y}{mr^2 \sin I} = A\psi = \ddot{\psi}(\cos^2 \dot{\phi} t) - \dot{\phi}\dot{\psi}\sin 2\dot{\phi} t \tag{29}$$

which once again is a damped harmonic oscillator equation with higher frequency modulating functions.

CONCLUSIONS

The consequences of the deeply embedded harmonic terms within the orbital equations of motion demonstrated above provide the opportunity for achieving trajectories for artificial satellites not widely considered at present. Two in particular will be discussed, and two suggestions about natural orbits will be mentioned as well.

1) Psi-Coupling

With the use of the substitution, $x = \cos 2\dot{\phi} t$ equation (29) may be restructured in the form

$$\ddot{\psi} + \frac{d}{dt}(-x\dot{\psi}) + A\psi = 0 \tag{30}$$

For certain values of $\dot{\psi}$ it has been shown that some higher period modulating frequencies may be ignored when considering one of two predominant frequencies in a two mode system.[5] This assumption reduces equation (30) to an undamped harmonic oscillator with the natural frequency of $\sqrt{A}$.

In general there is no restriction on equation (30) other than the physical possibility of applying torques to the orbit exactly proportional to the absolute angle of the argument of the ascending node. When the orbital period is much shorter than the precessional period, though, this restriction presents no great obstacle, since the Fourier components of impulses applied at the nodal line are also harmonic in the natural frequency. So in theory, at least $\sqrt{A}$ could be made precisely equal to twenty-four hours. Such a scheme will yield a psi-synchronous orbit (so named for the usual classical mechanics variable specifying precession angle). Twice every twenty-four hours the orbit retraces an exact footprint over the surface of the earth. These two turning longitudes can be chosen from the initial conditions to correspond with specific locations of communication or remote sensing importance in which the solar aspect angle is required to be the same over long periods of time. Furthermore, this synchronous solution is not dependent on the orbital radius, provided the proper steady state value of torque is applied.

2) Theta Coupling

In the same manner as psi-synchronous coupling is achieved a nutational frequency is excited as described by equations (17)–(22). Depending on the precession rate, the satellite orbit may be made to oscillate between two bounding latitudes over the Earth while covering specific portions of the same longitude. Figure 4 shows a generic pattern established by the angular momentum vector with this type of excitation. Essentially, the orbit wobbles about the equatorial plane with two orthonormal frequencies tailored by the character of the applied moments.

3) The Zonal Harmonics

As shown throughout this analysis, but specifically in equations (9), (10), (11), many coupling terms exist between the Eulerian variables of an orbit. Each of these variables will have sinusoidal terms as well as secular terms. In some instances, products of these sinusoidal terms lead to constant values with secondary sinusoidal terms of higher frequency. For example, if the precession and nutation rates are both sinusoidal the term $\dot{\psi}\sin I(\dot{\phi} + \dot{\psi}\cos I)$ in equation (9) will be

$$\bar{\dot{\psi}} \sin(at)\sin(bt)(\dot{\phi} - \bar{\dot{\psi}}\sin(at)\cos(\sin bt)) \tag{31}$$

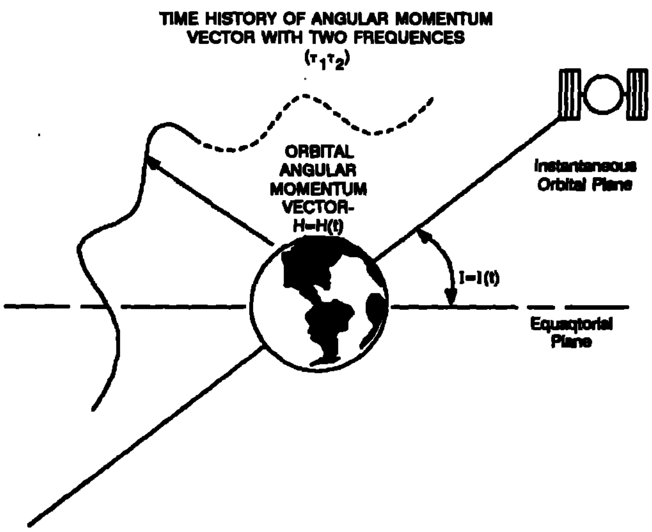

Figure 4 Time history of the angular momentum vector with two modal arm motion.

Sinusoidal functions with arguments that are in turn sinusoidal functions can be approximated closely by Bessel Series. Applying this technique to equation (31) gives Bessel functions with constants and overtones of the fundamental frequency at: $(a-b), 2(a-b), 3(a-b), \ldots$ The constant values accelerate the precessional velocity slightly. But the precessional velocity is used to calculate, by Legendre functions, both the zonal and tesseral harmonics of the Earth's geoid from observed satellite perturbations. The Legendre coefficients therefore may be incorporating part of a secular advance which is really due to the particular characteristic of the orbit rather than the nature of the earth's geoid.

A secondary application of this statement applies to natural satellites as well. The unaccounted for discrepancy between the precession of Mercury's orbit and rates predicted by Newton's law (even with modifications made by General Relativity) might be explained by this result.

4) Halley's Comet

While executing that portion of its trajectory near the periapsis, Halley's comet experiences a net thrust force. The force is caused by material being ejected from the surface of the comet as its constituent ices sublime when exposed to solar radiation. Theoretically, these forces should lie exactly in the orbital plane, producing no out-of-plane thrusts. Practically, however, many effects modify the actual thrust vector. These include: variations in the albedo of Halley's comet across its surface, interference of the solar wind, asphericity of the comet itself and asphericity of both the solar potential field and its luminosity with solar latitude.

The actual solar radiation received at the surface of the comet is a function of the cosine of the true anomaly while the vector component of the out-of-plane thrust is a function of the sine of the true anomaly and the inclination angle as shown in Figure 5. The product of the force term components then, is:

$$F = B_1 \sin\dot{\phi}t \cos\dot{\phi}t \sin I$$

which can be restated by trigonometric identity as

$$F = \frac{B_1}{2}(\sin 2\dot{\phi}t)\sin I \qquad (32)$$

Note that if equation (32) is substituted into equation (13) the forcing function occurs at a resonance frequency of the equation. At least to the extent of the small angle approximation certain conditions on the thrust vector could lead to large changes in the inclination angle. The same is true for total precession angle. Hence it may be possible Halley's comet began with an orbit quite different from its present day position. As a matter of speculation one may even wonder if its retrograde orbit, unusual among the other objects of the solar system, may not in some way have originated as a prograde orbit which over time experienced a large unstable change in inclination angle.

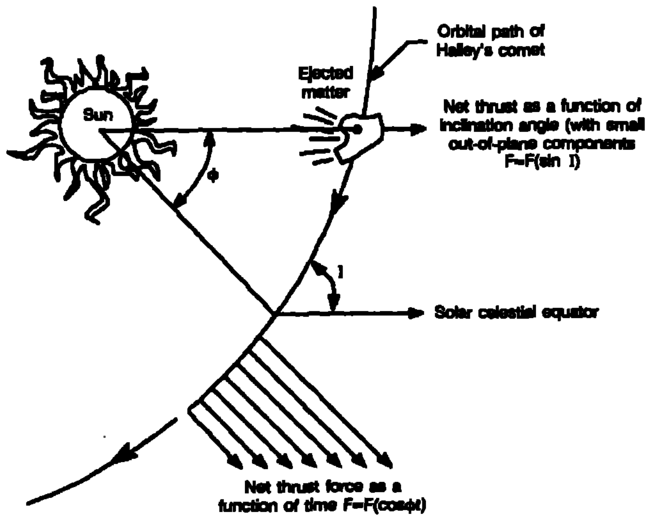

Figure 5 Orbital torque on Halley's comet as a function of two parameters, true anomaly and inclination angle.

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